Određeni integrali

Osobine određenih integrala su:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\int_a^b [f_1(x) + f_2(x) - f_3(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx - \int_a^b f_3(x)\,dx$

5. $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$, gdje je α konstanta

Određene integrale ćemo računati pomoću Njutn-Lejbnicove formule

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b = F(x)\Big|_a^b = F(b) - F(a)$$

gdje je $F'(x) = f(x)$

(#) Izračunati integrale

a) $\int_2^3 3x^2 dx$; b) $\int_0^4 (1+e^{\frac{x}{4}}) dx$; c) $\int_{-1}^7 \frac{dt}{\sqrt{3t+4}}$;

d) $\int_0^{\pi/2a} (x+3) \sin ax$

Rj.

a) $$\int_2^3 3x^2 dx = 3\int_2^2 x^2 dx = 3\cdot \frac{x^3}{3}\Big|_2^3 = x^3\Big|_2^3 = 3^3 - 2^3 = 27-8 = 19$$

b) $$\int_0^4 (1+e^{\frac{x}{4}}) dx = \int_0^4 dx + \int_0^4 e^{\frac{x}{4}} dx = \int_0^4 dx + 4\int_0^4 e^{\frac{x}{4}} d\left(\frac{x}{4}\right) =$$

$$= x\Big|_0^4 + 4\, e^{\frac{x}{4}}\Big|_0^4 = (4-0) + 4\left(e^{\frac{4}{4}} - e^{\frac{0}{4}}\right) = 4 + 4e - 4 = 4e$$

c) $$\int_{-1}^7 \frac{dt}{\sqrt{3t+4}} = \int_{-1}^7 (3t+4)^{-\frac{1}{2}} dt = \left| \begin{matrix} d(3t+4) = 3\,dt \\ dt = \frac{1}{3} d(3t+4) \end{matrix} \right| =$$

$$= \frac{1}{3}\int_{-1}^7 (3t+4)^{-\frac{1}{2}} d(3t+4) = \frac{2}{3}(3t+4)^{\frac{1}{2}}\Big|_{-1}^7 = \frac{2}{3}\left(\sqrt{25} - \sqrt{1}\right) = \frac{8}{3}$$

d) $$\int_0^{\frac{\pi}{2a}} (x+3)\sin ax\, dx = \left| \begin{matrix} u = x+3 & dv = \sin ax\, dx \\ du = dx & v = \frac{1}{a}\int \sin ax\, d(ax) = \frac{-1}{a}\cos ax \end{matrix} \right| =$$

$$= -\frac{1}{a}(x+3)\cos ax\Big|_0^{\frac{\pi}{2a}} + \frac{1}{a}\int_0^{\pi/2a} \cos ax\, dx = -\frac{1}{a}\left[\left(\frac{\pi}{2a}+3\right)\underbrace{\cos\frac{\pi}{2}}_{=0} - 3\underbrace{\cos 0}_{=1}\right] +$$

$$+ \frac{1}{a}\cdot\frac{1}{a}\int_0^{\pi/2a} \cos ax\, d(ax) = \frac{3}{a} + \frac{1}{a^2}\underbrace{\sin ax\Big|_0^{\frac{\pi}{2a}}}_{\sin\frac{\pi}{2} - \sin 0} = \frac{3}{a} + \frac{1}{a^2} = \frac{1+3a}{a^2}$$

Zadaci za vežbu

1. $\displaystyle\int_1^5 \frac{dx}{3x-2}$

2. $\displaystyle\int_0^1 \frac{dz}{(2z+1)^3}$

3. $\displaystyle\int_1^2 \frac{dt}{t^2+5t+4}$

4. $\displaystyle\int_0^2 \frac{x+3}{x^2+4}\,dx$

5. $\displaystyle\int_{-a}^{a} x\cos\frac{x}{a}\,dx$

6. $\displaystyle\int_0^{\pi} \cos\frac{x}{2}\cos\frac{3x}{2}\,dx$

7.* $\displaystyle\int_{-\pi}^{\pi} x\sin x\cos x\,dx$

8. $\displaystyle\int_1^{e} (1+\ln y)^2\,dy$

Rešenja:

1. $\dfrac{\ln 13}{3}$ 2. $\dfrac{2}{9}$ 3. $\dfrac{1}{3}\ln\dfrac{5}{4}$ 4. $\dfrac{3\pi}{8}+\dfrac{\ln 2}{2}$

5. 0 6. 0 7. $-\dfrac{\pi}{2}$ 8. $2e-1$

Zamena promenjivih u određenom integralu

$$\int_a^b f(x)\,dx = \left| \begin{matrix} x=\varphi(t) & x=a \Rightarrow a=\varphi(\alpha) \Rightarrow & t=\alpha \\ dx=\varphi'(t)\,dt & x=b \Rightarrow b=\varphi(\beta) \Rightarrow & t=\beta \end{matrix} \right|$$

$$= \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_\alpha^\beta F(t)\,dt$$

(#) Izračunati integrale

a) $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{1+3x}}$; b) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}}$; c) $\displaystyle\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}}$; d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Rj.

a) $$\int_0^5 \frac{x\,dx}{\sqrt{1+3x}} = \left| \begin{matrix} 1+3x=t^2 & 3\,dx = 2t\,dt \\ \sqrt{1+3x}=t & dx=\frac{2}{3}t\,dt \\ 3x=t^2-1 & \\ x=\frac{t^2-1}{3} & x\Big|_0^5 \Rightarrow t\Big|_1^4 \end{matrix} \right| = \int_1^4 \frac{\frac{t^2-1}{3}\cdot\frac{2}{3}t\,dt}{t} =$$

$$= \frac{2}{9}\int_1^4 (t^2-1)\,dt = \frac{2}{9}\left(\frac{t^3}{3}\Big|_1^4 - t\Big|_1^4\right) = \frac{2}{9}\left(\frac{1}{3}(64-1)-(4-1)\right) =$$

$$= \frac{2}{9}\left(\frac{63}{3}-3\right) = \frac{2}{9}(21-3) = 4$$

b) $$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}} = \left| \begin{matrix} e^x=t & e^{-x}=t^{-1}=\frac{1}{t} \\ x=\ln t & \\ dx=\frac{dt}{t} & x\Big|_{\ln 2}^{\ln 3} \Rightarrow t\Big|_2^3 \end{matrix} \right| = \int_2^3 \frac{\frac{dt}{t}}{t-\frac{1}{t}} = \int_2^3 \frac{\frac{dt}{t}}{\frac{t^2-1}{t}} =$$

$$= \int_2^3 \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Bigg|_2^3 = \frac{1}{2}\left(\ln\frac{2}{4}-\ln\frac{1}{3}\right) = \frac{1}{2}\cdot\ln\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\ln\frac{3}{2}}{2}$$

c)
$$\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}} = \left| \begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \qquad x\Big|_1^{\sqrt{3}} \Rightarrow t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ x^3 = 8\sin^3 t \\ \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = \sqrt{4(1-\sin^2 t)} \end{array} \right| =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(8\sin^3 t+1)\,2\cos t\,dt}{4\sin^2 t\sqrt{4\cos^2 t}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^3 t+1}{4\sin^2 t}\,dt =$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin t\,dt + \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dt}{\sin^2 t} = -2\cos t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{1}{4}\operatorname{ctg} t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= -2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{3} - \sqrt{3}\right) = \frac{7}{2\sqrt{3}} - 1$$

d)
$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2} = z & dx = \frac{2dz}{1+z^2} \\ \cos x = \frac{1-z^2}{1+z^2} & x\Big|_0^{\frac{\pi}{2}} \Rightarrow z\Big|_0^1 \end{array} \right| =$$

$$= \int_0^1 \frac{\frac{2dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = 2\int_0^1 \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{z}{\sqrt{3}}\Big|_0^1 =$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

(#) Dokazati da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

dok za neparnu f-ju $f(x)$ vrijedi $\int_{-a}^{a} f(x)\,dx = 0$.

Rj. Prvo rastavimo interval $[-a,a]$ na dva dijela $[-a,0]$ i $[0,a]$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \ldots(*)$$

Posmatrajmo sad $\int_{-a}^{0} f(x)\,dx$. Ako uvedemo smjenu $x=-z$ imamo da je $dx=-dz$ i $z_1=a$ za $x_1=-a$, $z_2=0$ za $x_2=0$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-z)\,dz = \int_{0}^{a} f(-z)\,dz = \left|\begin{matrix}\text{nove prom.}\\ x=z\end{matrix}\right| = \int_{0}^{a} f(-x)\,dx$$

Prema tome $(*)$ sad postaje

$$I = \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} [f(x)+f(-x)]\,dx$$

Za parnu f-ju znamo da $f(-x)=f(x)$ dok je za neparnu f-ju $f(-x)=-f(x)$. Prema tome

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & \text{ako je } f(x) \text{ parna f-ja} \\ 0, & \text{ako je } f(x) \text{ neparna f-ja} \end{cases}$$

(#) Znamo da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

dok za neparnu f-ju $f(x)$ vrijedi: $\int_{-a}^{a} f(x)\,dx = 0$.
Iskoristiti ovu osobinu i izračunati sljedeće integrale:

a) $\int_{-\sqrt{5}}^{\sqrt{5}} (3x-2x^5)\,dx$ b) $\int_{-\pi}^{\pi} \sin^7 2x\,dx$ c) $\int_{3}^{-3} t^8 \arcsin t\,dt$

d) $\int_{-2}^{2} \frac{x^5+7x^4+x^3-5x^2-2}{x^3+x}\,dx$

Rj:

a) $f(x) = 3x-2x^5$

$f(-x) = 3(-x)-2(-x)^5 = -3x+2x^5 = -(3x-2x^5) = -f(x)$

$$\int_{-\sqrt{5}}^{\sqrt{5}} (3x-2x^5)\,dx = \left| \begin{array}{c} \text{primijetimo} \\ \text{da je} \\ 3x-2x^5 \text{ neparna} \\ \text{f-ja} \end{array} \right| = 0$$

b) $f(x) = \sin^7 2x \Rightarrow f(-x) = (\sin 2(-x))^7 = (-\sin 2x)^7 = -\sin^7 2x = -f(x)$

Kako je $\sin^7 2x$ neparna f-ja $\int_{-\pi}^{\pi} \sin^7 2x\,dx = 0$

c) $\int_{3}^{-3} t^8 \arcsin t\,dt = 0$ ZAŠTO? OBJASNITI!

d) $$\int_{-2}^{2} \frac{x^5+7x^4+x^3-5x^2-2}{x^3+x}\,dx = \int_{-2}^{2} \frac{\overbrace{x^5+x^3}^{x^2(x^3+x)}}{x^3+x}\,dx + \int_{-2}^{2} \frac{7x^4-5x^2-2}{x^3-x}\,dx =$$

$$= \int_{-2}^{2} x^2\,dx + 0 = 2\int_{0}^{2} x^2\,dx = 2\,\frac{x^3}{3}\Big|_0^2 = \frac{16}{3}$$

Zadaci za vježbu

Izračunati integrale

1. $$\int_0^1 \frac{x^2\,dx}{(x+1)^4}$$ uvođenjem smjene $x+1=z$.

2. $$\int_0^{\ln 2} \sqrt{e^x-1}\,dx$$ uvođenjem smjene $\sqrt{e^x-1}=t$.

3. $$\int_{\sqrt{3}}^{\sqrt{7}} \frac{x^3\,dx}{\sqrt[3]{(x^2+1)^2}}$$ uvođenjem smjene $z=x^2+1$.

4. $$\int_1^e \frac{\sqrt[4]{1+\ln x}}{x}\,dx$$ uvođenjem smjene $t=1+\ln x$.

5. $$\int_{-3}^{3} x^2\sqrt{9-x^2}\,dx$$ uvođenjem smjene $x=3\cos\varphi$

6. $$\int_5^1 \frac{t\,dt}{\sqrt{5+4t}}$$

7. $$\int_0^{\frac{\pi}{4}} \frac{1+tg^2\varphi}{1+tg\,\varphi}\,d\varphi$$

8. $$\int_{\ln 3}^{0} \frac{1-e^x}{1+e^x}\,dx$$

9. $$\int_{-1}^{0} \frac{dx}{1+\sqrt[3]{x+1}}$$

10. * $$\int_0^8 \sqrt{\frac{x}{6-x}}\,dx$$

11. * $$\int_0^{\frac{\pi}{2}} \sin^3\varphi\sqrt{\cos\varphi}\,d\varphi$$

Rješenja:

1. $\frac{1}{24}$ 2. $\frac{4-\pi}{2}$ 3. 3 4. $0{,}8(2\sqrt[4]{2}-1)$ 5. $\frac{81\pi}{8}$ 6. $-\frac{17}{6}$

7. $\ln 2$ 8. $\ln\frac{4}{3}$ 9. $\frac{3}{2}(\ln 4-1)$ 10. $\frac{3(\pi-2)}{2}$
(uvodimo smjenu $x=6\sin^2 t$)

11. $8/21$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Računje određenih integrali i Smjena promjenjivih u određenim integralima)

1. $$\int_{\pi/6}^{\pi/3}\frac{dx}{\cos^2 x}=tg\,x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=tg\frac{\pi}{3}-tg\frac{\pi}{6}=\sqrt{3}-\frac{\sqrt{3}}{3}=\frac{2\sqrt{3}}{3}$$

2. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin x\,dx=-\cos x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=-\left(\cos\frac{\pi}{3}-\cos\frac{\pi}{4}\right)=-\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)=-\frac{1-\sqrt{2}}{2}$$

3. $$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{dx}{\sqrt{1-x^2}}=\arcsin x\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}=\arcsin\frac{\sqrt{3}}{2}-\arcsin\frac{1}{2}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

4. $$\int_a^b x^m\,dx=\frac{x^{m+1}}{m+1}\Big|_a^b=\frac{1}{m+1}\left(b^{m+1}-a^{m+1}\right)$$

5. $$\int_0^1(e^x-1)^4e^x\,dx=\left|\begin{array}{l} e^x-1=t \\ e^x dx=dt \\ x=0 \Rightarrow t=0 \\ x=1 \Rightarrow t=e-1\end{array}\right|=\int_0^{e-1}t^4\,dt=\frac{t^5}{5}\Big|_0^{e-1}=\frac{1}{5}(e-1)^5$$

6. $$\int_2^9\sqrt[3]{x-1}\,dx=\left|\begin{array}{l} x-1=t^3 \\ dx=3t^2dt \\ x=2 \Rightarrow t=1 \\ x=9 \Rightarrow t=2\end{array}\right|=\int_1^2\sqrt[3]{t^3}\cdot 3t^2\,dt=3\int_1^2 t^3\,dt=\frac{3}{4}t^4\Big|_1^2=$$
$$=\frac{3}{4}(16-1)=\frac{45}{4}$$

7. $$\int_0^{\ln 5}\frac{\sqrt{e^x-1}}{1+3e^{-x}}\cdot\frac{e^x}{e^x}\,dx=\int_0^{\ln 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx=\left|\begin{array}{l} e^x-1=t^2 \\ e^x dx=2t\,dt \\ x=0 \Rightarrow t=0 \\ x=\ln 5 \Rightarrow t=2 \\ e^x=t^2+1\end{array}\right|=\int_0^2\frac{\sqrt{t^2}\cdot 2t}{t^2+1+3}\,dt$$
$$=2\int_0^2\frac{t^2+4-4}{t^2+4}\,dt=2\int_0^2 dt-2\int_0^2\frac{4}{t^2+4}\,dt=2t\Big|_0^2-8\cdot\frac{1}{2}\,\mathrm{arctg}\,\frac{t}{2}\Big|_0^2=$$
$$=4-4\left(\mathrm{arctg}\,1-\mathrm{arctg}\,0\right)=4-4\cdot\frac{\pi}{4}=4-\pi$$

Osobine određenih integrala

a) $\int_a^a f(x)\,dx = 0$

b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c) $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$

d) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

e) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall x$

8. $$\int_0^{\sqrt{7}} \frac{dx}{7+x^2} = \frac{1}{\sqrt{7}} \operatorname{arctg} \frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}\left(\operatorname{arctg}\frac{\sqrt{7}}{\sqrt{7}} - \operatorname{arctg}\frac{0}{\sqrt{7}}\right) = \frac{1}{\sqrt{7}}\cdot\frac{\pi}{4} = \frac{\sqrt{7}\pi}{28}$$

9. $$\int_0^{1/2} \sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \\ x=0 \Rightarrow \sin t = 0 \Rightarrow t=0 \\ x=\frac{1}{2} \Rightarrow \sin t = \frac{1}{2} \Rightarrow t=\frac{\pi}{6} \end{array} \right| = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\,\cos t\,dt =$$

kako je $\sin^2 t + \cos^2 t = 1$

$$= \int_0^{\pi/6} \sqrt{\cos^2 t}\,\cos t\,dt = \int_0^{\pi/6} \cos^2 t\,dt = \frac{1}{2}\int_0^{\pi/6}(1+\cos 2t)\,dt = \frac{1}{2}t\Big|_0^{\frac{\pi}{6}} + \frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{6}}$$

$1 = \cos^2 t + \sin^2 t$

$\cos 2t = \cos^2 t - \sin^2 t$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{2\pi + 3\sqrt{3}}{24}$$

10. $$\int_0^4 \frac{dx}{1+\sqrt{2x+1}}$$

uputa: smjena $2x+1 = t^2$

rj. $2 - \ln 2$

11. $$\int_0^1 \frac{dx}{\sqrt{2-x^2+x}}$$

uputa: $-x^2+x+2 = \ldots$

$\ldots = \frac{9}{4} - (x-\frac{1}{2})^2$

$x - 1 = \frac{3}{2}t$

rj. $2\arcsin\frac{1}{3}$

12. $$\int_1^e x\ln x\,dx = \left| \begin{array}{ll} u = \ln x & dv = x\,dx \\ du = \frac{dx}{x} & v = \frac{x^2}{2} \end{array} \right| = \frac{1}{2}x^2\ln x\Big|_1^e - \frac{1}{2}\int_1^e x^2 \frac{dx}{x} =$$

$$= \frac{1}{2}(e^2\ln e - 1^2 \ln 1) - \frac{1}{2}\int_1^e x\,dx = \frac{1}{2}e^2 - \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_1^e =$$

$$= \frac{1}{2}e^2 - \frac{1}{4}(e^2-1) = \frac{1}{4}e^2 + \frac{1}{4}$$

Izračunati integral $\displaystyle\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5}\,dx$.

Rj. $x-4\sqrt{x}+5 = x-2\cdot\sqrt{x}\cdot 2+4+1 = (\sqrt{x}-2)^2+1$

$$\int_1^4 \frac{\sqrt{x}+2}{(\sqrt{x}-2)^2+1}\,dx = \left| \begin{array}{l} x=t^2 \\ dx = 2t\,dt \\ x=1 \Rightarrow t=1 \\ x=4 \Rightarrow t=2 \end{array} \right| = \int_1^2 \frac{t+2}{(t-2)^2+1}\cdot 2t\,dt =$$

$$=2\int_1^2 \frac{t^2+2t}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2+2t-6t+6t+5-5}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2-4t+5}{t^2-4t+5}\,dt +$$

$$+2\int_1^2 \frac{6t-5}{t^2-4t+5}\,dt = 2\int_1^2 dt + 2\cdot 3\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}$$

$$\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4+4-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt + \frac{7}{3}\int_1^2 \frac{dt}{t^2-4t+5}$$

$$\int_1^2 dt = t\Big|_1^2 = 2-1 = 1, \quad \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt = \left| \begin{array}{l} t^2-4t+5 = s \\ (2t-4)\,dt = ds \\ t=1 \Rightarrow s=2 \\ t=2 \Rightarrow s=1 \end{array} \right| = \int_2^1 \frac{ds}{s} = \ln|s|\Big|_2^1$$

$$= \ln 1 - \ln 2 = -\ln 2$$

$$\int_1^2 \frac{dt}{t^2-4t+5} = \int_1^2 \frac{dt}{(t-2)^2+1} = \left| \begin{array}{l} t-2=s \\ dt=ds \\ t=1 \Rightarrow s=-1 \\ t=2 \Rightarrow s=0 \end{array} \right| = \int_{-1}^0 \frac{ds}{s^2+1} = \operatorname{arctg} s\Big|_{-1}^0 = 0-\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} = 2\cdot 1 + 6\left(-\ln 2 + \frac{7}{3}\cdot\frac{\pi}{4}\right) = 2 - 6\ln 2 + \frac{7\pi}{2} \approx 8{,}8367$$

traženo rešenje

(#) Izračunati integral $\int_0^{\frac{\pi}{4}} \sin^5 x \cos^7 x \, dx$

Rj. $\int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^7 x \, dx = \int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^6 x \cdot \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \\ x=0 \Rightarrow t=0 \\ x=\frac{\pi}{4} \Rightarrow t=\frac{\sqrt{2}}{2} \\ \cos^6 x = (\cos^2 x)^3 = \\ = (1-\sin^2 x)^3 = (1-t^2)^3 \end{array} \right| =$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1-t^2)^3 \, dt =$$

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \end{array}$$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1-3t^2+3t^4-t^6) \, dt = \int_0^{\frac{\sqrt{2}}{2}} (t^5-3t^7+3t^9-t^{11}) \, dt =$$

$$= \frac{1}{6} t^6 \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{3}{8} t^8 \Big|_0^{\frac{\sqrt{2}}{2}} + \frac{3}{10} t^{10} \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{1}{12} t^{12} \Big|_0^{\frac{\sqrt{2}}{2}} =$$

$$= \frac{1}{\cancel{6}_3} \cdot \frac{\cancel{8}}{\cancel{64}_{16}} - \frac{3}{\cancel{8}_1} \cdot \frac{\cancel{16}}{\cancel{256}_{128}} + \frac{3}{\cancel{10}_5} \cdot \frac{\cancel{32}}{\cancel{1024}_{64}} - \frac{1}{\cancel{12}_3} \cdot \frac{\cancel{64}}{\cancel{4096}_{256}} =$$

$$= \frac{1}{\underbrace{3 \cdot 16}_{3 \cdot 2^4}} - \frac{3}{\underbrace{128}_{2^7}} + \frac{3}{\underbrace{5 \cdot 64}_{5 \cdot 2^6}} - \frac{1}{\underbrace{3 \cdot 256}_{3 \cdot 2^8}} = \frac{5 \cdot 2^4 - 3 \cdot 3 \cdot 5 \cdot 2 + 3 \cdot 3 \cdot 2^2 - 5}{3 \cdot 5 \cdot 2^8}$$

$$= \frac{80-90+36-5}{3 \cdot 5 \cdot 2^8} = \frac{\cancel{21}^{7}}{\cancel{3}_1 \cdot 5 \cdot 2^8} = \frac{7}{1280}$$ traženo rješenje

(#) Izračunati integral $\displaystyle\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx$

Rj.

Metoda Ostrogradskog:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = (ax^2+bx+c)\sqrt{x^2+3} + \lambda \int \frac{dx}{\sqrt{x^2+3}} \quad \Big/ \frac{d}{dx}$$

$$\frac{2x^3-7x+4}{\sqrt{x^2+3}} = (2ax+b)\sqrt{x^2+3} + (ax^2+bx+c)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}} + \frac{\lambda}{\sqrt{x^2+3}} \quad \Big/\cdot\sqrt{x^2+3}$$

$$2x^3-7x+4 = (2ax+b)(x^2+3) + (ax^2+bx+c)x + \lambda$$

$$2x^3-7x+4 = 2ax^3+bx^2+6ax+3b+ax^3+bx^2+cx+\lambda$$

x^3: $2a+a=2 \Rightarrow 3a=2$, $a=\frac{2}{3}$

x: $6a+c=-7 \Rightarrow 6\cdot\frac{2}{3}+c=-7$

$4+c=-7$

$c=-11$

x^2: $b+b=0 \Rightarrow b=0$

x^0: $3b+\lambda=4$

$\lambda=4$

Prema tome:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \left(\frac{2}{3}x^2-11\right)\sqrt{x^2+3} + 4\int\frac{dx}{\sqrt{x^2+3}} =$$

$$= \frac{2}{3}x^2\sqrt{x^2+3} - 11\sqrt{x^2+3} + 4\ln\left|x+\sqrt{x^2+3}\right| + C$$

Prema tome

$$\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \frac{2}{3}x^2\sqrt{x^2+3}\Big|_{-1}^{1} - 11\sqrt{x^2+3}\Big|_{-1}^{1} + 4\ln\left|x+\sqrt{x^2+3}\right|\Big|_{-1}^{1} =$$

$$= \frac{2}{3}(2-2) - 11(2-2) + 4\left(\ln|1+2| - \ln|-1+2|\right) =$$

$$= 4(\ln 3 - \ln 1) = 4\ln 3$$ traženi rezultat

\# Izračunati: $\displaystyle\int_3^4 \frac{6x+8}{x^2+x-6}\,dx$

Rj.

$$\frac{6x+8}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3} \quad /(x-2)(x+3)$$

$D = 1+24 = 25$

$x_{1,2} = \dfrac{-1 \pm 5}{2}$

$x_1 = -3 \quad x_2 = 2$

$x^2+x-6 = (x-2)(x+3)$

$$6x+8 = A(x+3) + B(x-2)$$

$$6x+8 = (A+B)x + (3A-2B)$$

$$\begin{aligned} A+B &= 6 \quad /\cdot 2 \\ 3A-2B &= 8 \\ \hline 2A+2B &= 12 \\ + \; 3A-2B &= 8 \\ \hline 5A &= 20 \\ A &= 4 \end{aligned}$$

$$\begin{aligned} A+B &= 6 \\ 4+B &= 6 \\ B &= 2 \end{aligned}$$

$$\int \frac{6x+8}{x^2+x-6}\,dx = \int\left(\frac{4}{x-2} + \frac{2}{x+3}\right)dx = 4\int\frac{dx}{x-2} + 2\int\frac{dx}{x+3} =$$

$$= 4\ln|x-2| + 2\ln|x+3| + C$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = 4\ln|x-2|\Big|_3^4 + 2\ln|x+3|\Big|_3^4 = 4(\ln 2 - \underbrace{\ln 1}_{=0}) +$$

$$+ 2(\ln 7 - \ln 6) = 4\ln 2 + 2\ln\frac{7}{6} = \ln 2^4 + \ln\left(\frac{7}{6}\right)^2$$

$$= \ln\frac{7^2}{2^2\cdot 3^2}\cdot 2^4 = \ln\frac{49\cdot 4}{9} = \ln\frac{196}{9}$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = \ln\frac{196}{9} \qquad \text{traženo rješenje}$$

Izračunati integral $I=\int_0^1 \arcsin\frac{x}{2}\,dx$.

Rj.

$$\int \arcsin\frac{x}{2}\,dx = \left| \begin{array}{l} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2dt \end{array} \right| = 2\int \arcsin t\,dt = \left| \begin{array}{ll} u=\arcsin t & dv=dt \\ du=\frac{dt}{\sqrt{1-t^2}} & v=t \end{array} \right| =$$

$$= 2\left(t\arcsin t - \int \frac{t}{\sqrt{1-t^2}}\,dt\right) = 2t\arcsin t - \int \frac{2t\,dt}{\sqrt{1-t^2}} \overset{(**)}{=\!=}$$

$$\int \frac{-2t\,dt}{\sqrt{1-t^2}} = \left| \begin{array}{l} 1-t^2=s \\ -2t\,dt=ds \end{array} \right| = \int \frac{ds}{\sqrt{s}} = \int s^{-\frac{1}{2}}ds = \frac{s^{\frac{1}{2}}}{\frac{1}{2}}+C = 2\sqrt{s}+C =$$

$$= 2\sqrt{1-t^2}+C$$

$$\overset{(**)}{=\!=} 2t\arcsin t + 2\sqrt{1-t^2}+C = x\arcsin\frac{x}{2} + 2\sqrt{1-\frac{x^2}{4}}+C$$

$$\int_0^1 \arcsin\frac{x}{2}\,dx = x\arcsin\frac{x}{2}\Big|_0^1 + 2\sqrt{1-\frac{x^2}{4}}\Big|_0^1 = \arcsin\frac{1}{2} + \left(2\sqrt{1-\frac{1}{4}}-2\right) =$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{2} - 2 = \frac{\pi}{6}+\sqrt{3}-2$$

Nepravi integrali

Nepravi integral u granicama od a do $+\infty$ je oblika

$$I = \int_a^{+\infty} f(x)\,dx$$

Rješavamo ga na sljedeći način:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

Ako postoji

$$\lim_{b\to\infty} \int_a f(x)\,dx .$$

kažemo da integral konvergira ili da postoji nepravi integral, a ako limes ne postoji (kao realan broj), kažemo da integral divergira ili da nepravi integral ne postoji.

1. Izračunati:

a) $$\int_1^{+\infty} \frac{dx}{x^4} = \lim_{a\to+\infty} \int_1^a \frac{dx}{x^4} = \lim_{a\to+\infty} \int_1^a x^{-4}\,dx = \lim_{a\to+\infty} \frac{x^{-3}}{-3}\Big|_1^a =$$

$$= -\frac{1}{3} \lim_{a\to\infty} \frac{1}{x^3}\Big|_1^a = -\frac{1}{3} \lim_{a\to\infty} \Big(\overset{\nearrow 0}{\frac{1}{a^3}} - 1\Big) = \Big(-\frac{1}{3}\Big)(-1) = \frac{1}{3}$$

b) $$\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b x^{-\frac{1}{2}}\,dx = \lim_{b\to\infty} \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big|_1^b = 2 \lim \sqrt{x}\Big|_1^b$$

$$= 2 \lim_{b\to\infty} (\sqrt{b} - 1) = +\infty ,$$ integral divergira

c) $$\int_0^{+\infty} e^{-ax}\,dx = \lim_{t\to\infty} \int_0^t e^{-ax}\,dx = \left| \begin{matrix} -ax = s & x=0 \Rightarrow s=0 \\ -a\,dx = ds & x=t \Rightarrow s=-at \\ dx = -\frac{1}{a}\,ds & \end{matrix} \right| =$$

$(a>0)$

$$= \lim_{t\to\infty} \int_0^{-at} e^s\Big(-\frac{1}{a}\Big)\,ds = -\frac{1}{a} \lim_{t\to\infty} e^s\Big|_0^{-at} = -\frac{1}{a} \lim_{t\to\infty} \Big(\overset{\nearrow 0}{e^{-at}} - 1\Big) = \frac{1}{a}$$

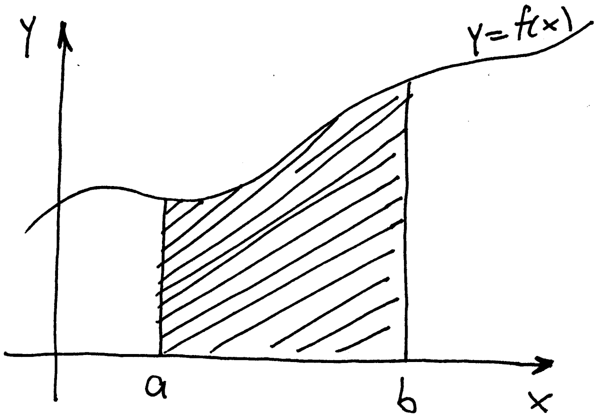
d) $\displaystyle\int_2^{+\infty} \frac{\ln x}{x}\,dx$ Rj. divergira

e) $\displaystyle\int_1^{+\infty} \frac{dx}{x^2(x+1)}$ Rj. $1-\ln 2$

f) $\displaystyle\int_0^{+\infty} x e^{-x^2}\,dx$ Rj. $\frac{1}{2}$

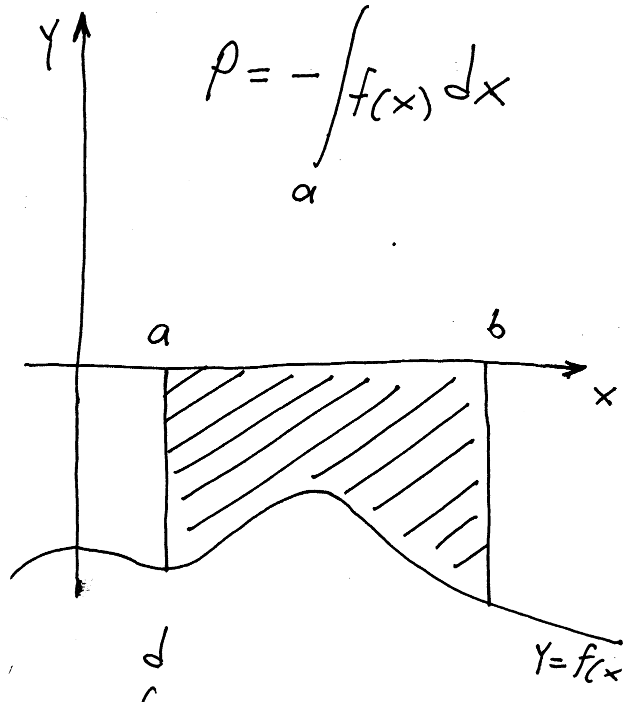
Primjena određenog integrala

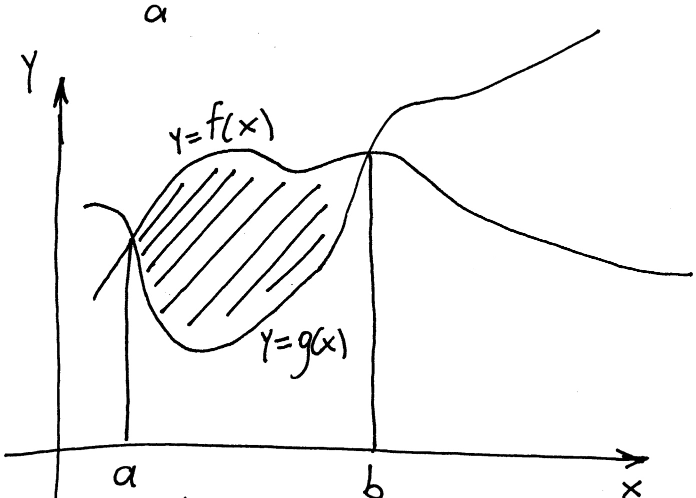
I Izračunavanje površine ravne figure



$$P=\int_a^b f(x)\,dx$$

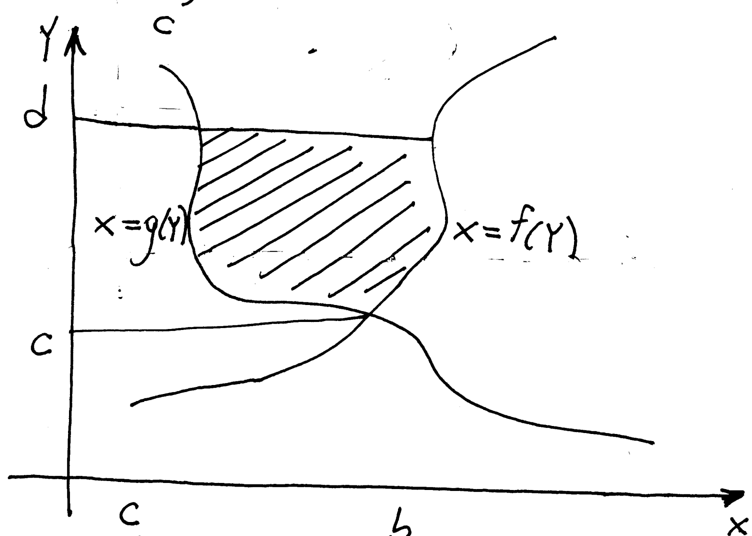
$$P=-\int_a^b f(x)\,dx$$

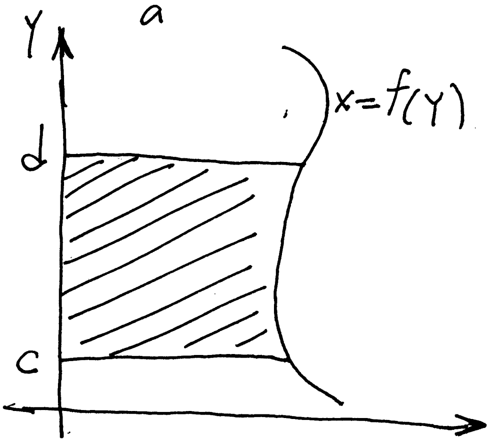




$$P=\int_a^b [f(x)-g(x)]\,dx$$

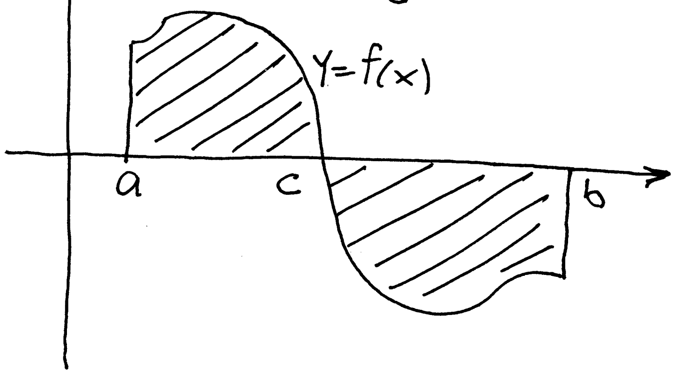
$$P=\int_c^d [f(y)-g(y)]\,dy$$





$$P=\int_c^d f(y)\,dy$$

$$P=\int_a^c f(x)\,dx+\left|\int_c^b f(x)\,dx\right|$$



1. Izračunati površinu ravne figure koja je ograničena linijama $y=4-(x-2)^2$ i $y=0$.

Rj.

$y=4-(x-2)^2$

$y=4-(x^2-4x+4)$

$y=-x^2+4x$

$y=-x(x-4)$

Nule $A(0,0)$ i $B(4,0)$

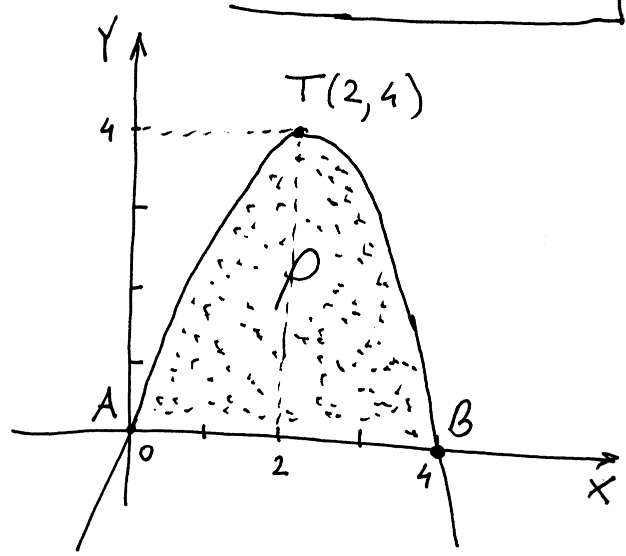
Kriva $y=ax^2+bx+c$ ima grafik u obliku parabole.
Tjeme parabole $T\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

za $a>0$ ∪ za $a<0$ ∩

$-\frac{b}{2a}=-\frac{4}{2\cdot(-1)}=2$

$\frac{4ac-b^2}{4a}=\frac{0-16}{-4}=4$

Tjeme parabole $y=4-(x-2)^2$ je u tački $(2,4)$.



$$P=\int_0^4(-x^2+4x)\,dx=\int_0^4(-x^2)\,dx+\int_0^4 4x\,dx=-\frac{x^3}{3}\Big|_0^4+4\cdot\frac{x^2}{2}\Big|_0^4=-\frac{1}{3}(4^3-0^3)$$

$$+2(4^2-0^2)=-\frac{1}{3}\cdot 64+32=\frac{96}{3}-\frac{64}{3}=\frac{32}{3}$$

2. Izračunati površinu ravne figure koja je ograničena krivom $y=x^2-4x+3$ i pravama $y=0$, $x=0$ i $x=2$.

Rj. $y=x^2-4x+3$

$D=16-12=4$

$x_{1,2}=\frac{4\pm 2}{2}$

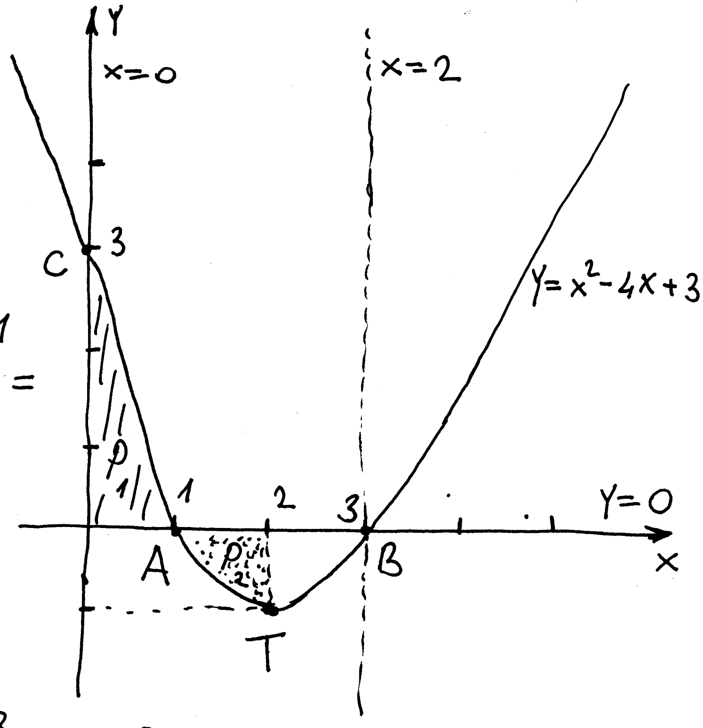
Nule krive $A(1,0)$ i $B(3,0)$

$-\frac{b}{2a}=-\frac{-4}{2}=2$

$\frac{4ac-b^2}{4a}=\frac{12-16}{4}=-1$

Tjeme krive $y=x^2-4x+3$ je u tački $T(2,-1)$.

$C(0,3)$ je tačka presjeka krive sa Y-osom



$P = P_1 + P_2$

$$P_1 = \int_0^1 (x^2-4x+3)\,dx = \frac{x^3}{3}\Big|_0^1 - 4\cdot\frac{x^2}{2}\Big|_0^1 + 3x\Big|_0^1 =$$

$$= \frac{1}{3}(1-0) - 2(1-0) + 3(1-0) =$$

$$= \frac{1}{3} + 1 = \frac{4}{3}$$

$$P_2 = -\int_1^2 (x^2-4x+3)\,dx = -\left(\frac{x^3}{3}\Big|_1^2 - 4\cdot\frac{x^2}{2}\Big|_1^2 + 3x\Big|_1^2\right) = -\left(\frac{1}{3}(8-1) - 2(4-1) + 3\cdot 1\right)$$

$$= -\left(\frac{7}{3} - 6 + 3\right) = -\left(-\frac{2}{3}\right) = \frac{2}{3}$$

$P = \frac{4}{3} + \frac{2}{3} = 2$ tražena površina ravne figure.

3. Izračunati površinu ravne figure kojeg čine parabola $y = x^2 - 2x + 2$ i prava $x + 2y - 9 = 0$.

Rj.

prava $x + 2y - 9 = 0$

$2y = -x + 9$

$y = -\frac{1}{2}x + \frac{9}{2}$

prava prolazi kroz tačke $A(0, \frac{9}{2})$ i $B(9, 0)$.

$y = x^2 - 2x + 2$

$D = 4 - 8 = -4 < 0$

kriva ne siječe x-o
- nema nula

$x = 0 \Rightarrow y = 2$

$C(0,2)$ je presjek krive sa Y-osom

$$-\frac{b}{2a} = -\frac{-2}{2} = 1$$

$$\frac{4ac - b^2}{4a} = \frac{8-4}{4} = 1$$

$T(1,1)$ je tjeme parabole

Trebamo naći još tačke presjeka prave i parabole.

$$\frac{\begin{array}{l} y=x^2-2x+2 \\ x+2y-9=0 \end{array}}{\frac{\begin{array}{l} y=x^2-2x+2 \\ x=-2y+9 \end{array}}{y=(-2y+9)^2-2(-2y+9)+2}}$$

$$y=4y^2-36y+81+4y-18+2$$

$$4y^2-33y+65=0$$

$$D=33^2-16\cdot 65=49$$

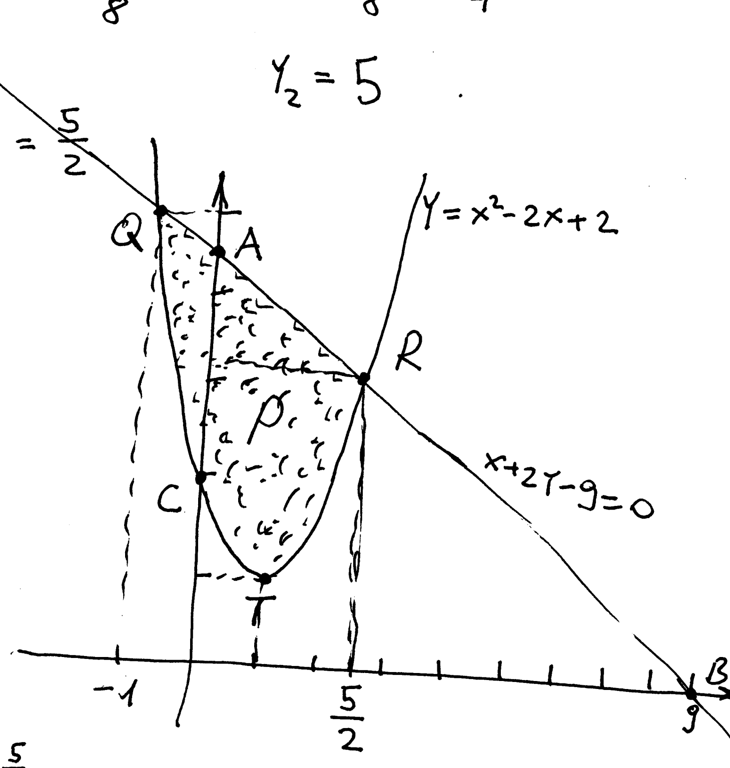
$$y_{1,2}=\frac{33\pm 7}{8} \qquad y_1=\frac{26}{8}=\frac{13}{4}$$

$$y_2=5$$

$$y_1=\frac{13}{4} \Rightarrow x=-2\cdot\frac{13}{4}+9=-\frac{13}{2}+\frac{18}{2}=\frac{5}{2}$$

$$y_2=5 \Rightarrow x=-2\cdot 5+9=-1$$

Tačke presjeka prave i parabole su $R\left(\frac{5}{2},\frac{13}{4}\right)$ i $Q(-1,5)$



$$P=\int_{-1}^{\frac{5}{2}}\left[\left(-\frac{1}{2}x+\frac{9}{2}\right)-(x^2-2x+2)\right]dx$$

$$\int_{-1}^{\frac{5}{2}}\left(-\frac{1}{2}x+\frac{9}{2}\right)dx=-\frac{1}{2}\cdot\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}}+\frac{9}{2}x\Big|_{-1}^{\frac{5}{2}}=-\frac{1}{4}\left(\frac{25}{4}-1\right)+\frac{9}{2}\left(\frac{5}{2}-(-1)\right)$$

$$=-\frac{1}{4}\cdot\frac{21}{4}+\frac{9}{2}\cdot\frac{7}{2}=\frac{231}{16}$$

$$\int_{-1}^{\frac{5}{2}}(x^2-2x+2)\,dx=\frac{x^3}{3}\Big|_{-1}^{\frac{5}{2}}-2\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}}+2x\Big|_{-1}^{\frac{5}{2}}=\frac{1}{3}\left(\frac{125}{8}-(-1)\right)-\left(\frac{25}{4}-1\right)+$$

$$+2\left(\frac{5}{2}-(-1)\right)=\frac{1}{3}\cdot\frac{133}{8}-\frac{21}{4}+2\cdot\frac{7}{2}=\frac{133}{24}+\frac{7}{4}=\frac{175}{24}$$

$$P=\frac{231}{16}-\frac{175}{24}=\frac{231}{4\cdot 4}-\frac{175}{6\cdot 4}=\frac{686}{96}=\frac{343}{48}$$

4. Izračunati površinu ravne figure koja je ograničena krivom $y^2=2x+1$ i pravom $y=2x-1$.

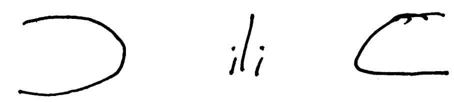
Rj. prava $y=2x-1$ prolazi kroz tačke $A(0,-1)$ i $B\left(\frac{1}{2},0\right)$.

$y^2 = 2x + 1$

$2x = y^2 - 1$

$x = \frac{1}{2}y^2 - \frac{1}{2}$

Kriva oblika $x = ay^2 + bx + c$ ima grafik u obliku parabole



ili

$a < 0$ $\qquad$ $a > 0$

Tjeme krive se traži po formuli $T\left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$x = 0 \Rightarrow y^2 = 1$

$A(0,-1)$; $B(0,1)$ su tačke presjeka f-je sa y-osom

$C(-\frac{1}{2}, 0)$ je nula f-je

$D = 1 \qquad -\frac{D}{4a} = -\frac{1}{4 \cdot \frac{1}{2}}$

$-\frac{b}{2a} = -\frac{0}{2 \cdot \frac{1}{2}} = 0$

$T(-\frac{1}{2}, 0)$ je tjeme parabole

Tražimo još tačke presjeka krive i prave

$y = 2x - 1$

$\underline{y^2 = 2x + 1}$

$(2x-1)^2 = 2x + 1$

$4x^2 - 4x + 1 - 2x - 1 = 0$

$4x^2 - 6x = 0$

$2x(2x - 3) = 0$

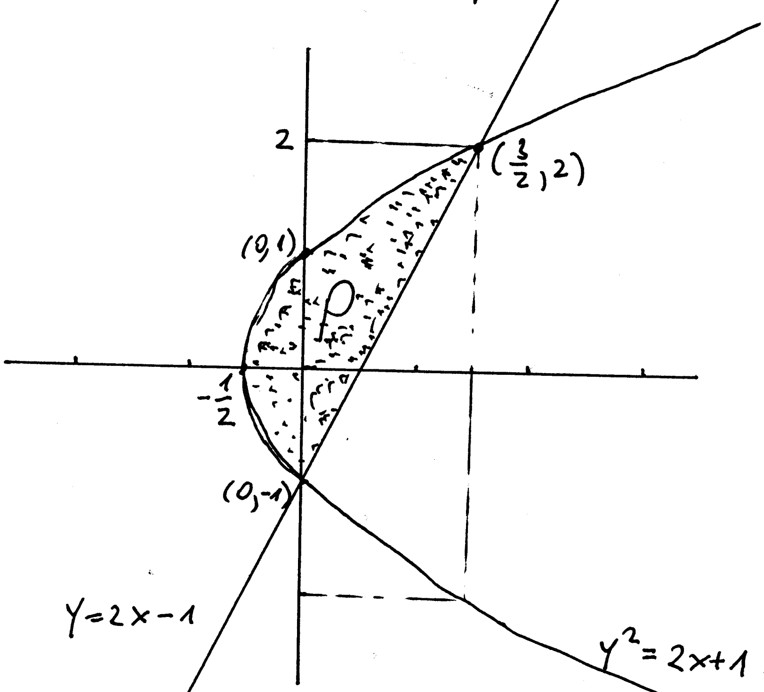
$x = 0 \quad \vee \quad x = \frac{3}{2}$

za $x = 0$

$\Downarrow$

$y = -1$

$D(0,-1)$ i $E(\frac{3}{2}, 2)$ su tačke presjeka krive i prave



$y = 2x - 1 \Rightarrow x = \frac{1}{2}y + \frac{1}{2}$

$y^2 = 2x + 1 \Rightarrow x = \frac{1}{2}y^2 - \frac{1}{2}$

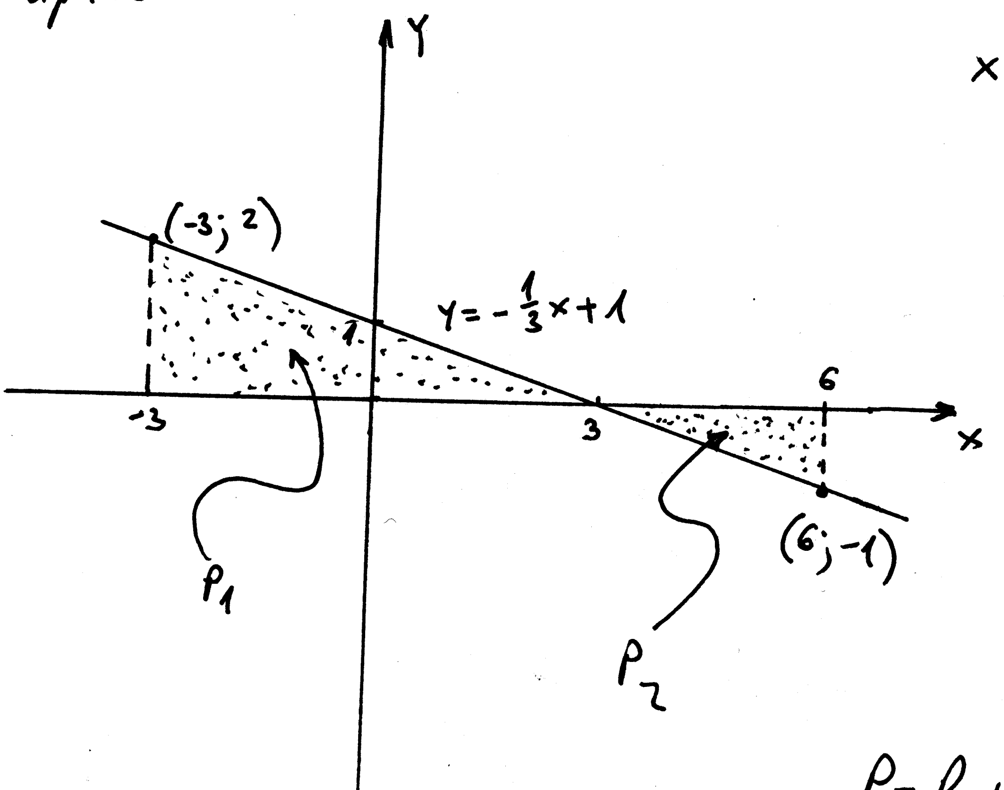
$$P = \int_{-1}^{2} \left[\frac{1}{2}y + \frac{1}{2} - \left(\frac{1}{2}y^2 - \frac{1}{2}\right)\right] dy = \frac{1}{2}\int_{-1}^{2} (y + 1 - y^2 + 1)\, dy = \frac{1}{2}\int_{-1}^{2} (-y^2 + y + 2)\, dy =$$

$$= \frac{1}{2} \cdot \left[\left(-\frac{y^3}{3}\right)\Big|_{-1}^{2} + \frac{y^2}{2}\Big|_{-1}^{2} + 2y\Big|_{-1}^{2}\right] = \frac{1}{2}\left[-\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2(2+1)\right] =$$

$$= \frac{1}{2}\left(-3 + \frac{3}{2} + 6\right) = \frac{1}{2} \cdot \frac{9}{2} = \frac{9}{4}$$ tražena površina

(#) Primenom određenog integrala odrediti površinu figure koju ograničava x-osa zajedno sa linijama $x+3y-3=0$, $x=-3$ i $x=6$.

Rj.-uputе



$x+3y-3=0$

$-3y=x-3 \quad /:(-3)$

$y=-\frac{1}{3}x+1$

$P=P_1+P_2$

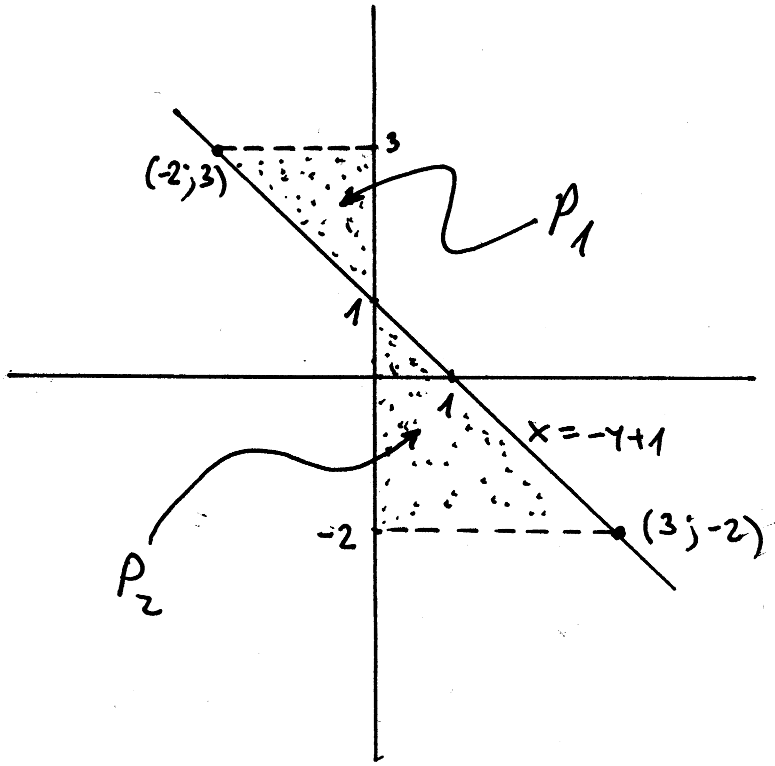
$$P_1=\int_{-3}^{3}\left(-\frac{1}{3}x+1\right)dx=\ldots=6$$

$$P_2=\left|\int_{3}^{6}\left(-\frac{1}{3}x+1\right)dx\right|=\ldots=+\frac{3}{2}$$

$$P=6+\frac{3}{2}=\frac{15}{2}$$

(#) Primjerom određenog integrala odrediti površinu figure koju ograničava y-osa zajedno sa linijama $x+y-1=0$, $y=3$ i $y=-2$.

Rj.-upute



$$x+y-1=0$$

$$x=-y+1$$

$$P=P_1+P_2$$

$$P_1=\left|\int_1^3(-y+1)\,dy\right|=\ldots=2$$

$$P_2=\int_{-2}^{1}(-y+1)\,dy=\ldots=\frac{9}{2}$$

$$P=2+\frac{9}{2}=\frac{13}{2}$$

Na parabolu $y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i y-osa.

Rj.

$y=-x^2+1$ parabola izgleda

$y=1-x^2$

$y(0)=1$

$(0,1)$ je presjek sa y-osom

$1-x^2=0$

$x^2=1$

$x_{1,2}=\pm 1$

$(-1,0)$ i $(1,0)$ su nule f-je

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$-\frac{b}{2a}=-\frac{0}{2\cdot(-1)}=0$

$D=0-4(-1)(1)=4$

$-\frac{D}{4a}=-\frac{4}{4\cdot(-1)}=1$

$T(0,1)$

$y-y_1=y'(x_1)(x-x_1)$

jednačina tangente u tački (x_1,y_1)

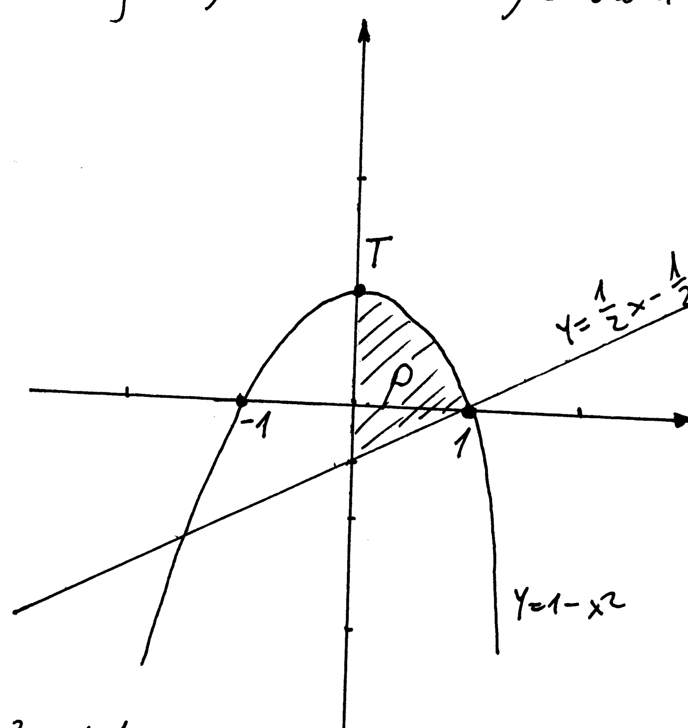
$y-y_1=-\frac{1}{y'(x_1)}(x-x_1)$ jednačina normale u tački (x_1,y_1)

$y'=-2x$ presjek parabole i pozitivnog dijela x-ose je tačka $(1,0)$

$y'(1)=-2$

$y-0=-\frac{1}{-2}(x-1)$

$y=\frac{1}{2}x-\frac{1}{2}$ jednačina normale u tački $(1,0)$



$$P=\int_0^1\left[(1-x^2)-\left(\frac{1}{2}x-\frac{1}{2}\right)\right]dx=$$

$$=\int_0^1\left(-x^2-\frac{1}{2}x+\frac{3}{2}\right)dx=-\frac{1}{3}x^3\Big|_0^1-\frac{1}{4}x^2\Big|_0^1+\frac{3}{2}x\Big|_0^1$$

$$=-\frac{1\cdot 4}{3\cdot 4}-\frac{1\cdot 3}{4\cdot 3}+\frac{3}{2}=\frac{3\cdot 6}{2\cdot 6}-\frac{7}{12}=\frac{18-7}{12}=\frac{11}{12}$$

$P=\frac{11}{12}$ tražena površina

(#) Izračunati površinu koju gradi kriva $y=x^2+x-6$ zajedno sa svojim tangentama povučenim na tu krivu u nultačkama krive.

R: $y=x^2+x-6$

$D=1+24=25$

$y=(x-2)(x+3)$

$x_1=2 \quad x_2=-3$

$(2,0)$ i $(-3,0)$ su nule f-je

$f(0)=-6$

$(0,-6)$ je tačka presjeka f-je sa y-osom

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ je tjeme f-je

$-\frac{b}{2a}=-\frac{1}{2}, \quad -\frac{D}{4a}=-\frac{25}{4}=-6\frac{1}{4}$

$T\left(-\frac{1}{2}, -6\frac{1}{4}\right)$

$a>0$

f-ja je $\cup$ oblika

$y-y_1=k(x-x_1)$ jednačina prave kroz tačku (x_1,y_1) i koeficijentom k

u slučaju tangente $k=y'(x_1)$

$y'=2x+1$

$(-3,0), \quad y'(-3)=-5$

$y-0=-5(x+3)$

$y=-5x-15$ jednačina tangente na krivu y u tački $(-3,0)$

$(2,0), \quad y'(2)=5$

$y-0=5(x-2)$

$y=5x-10$ jednačina tangente na krivu y u tački $(2,0)$

presjek pravih:

$y=-5x-15 \quad (1)$

$y=5x-10 \quad (2)$

$(1)+(2): \quad 2y=-25$

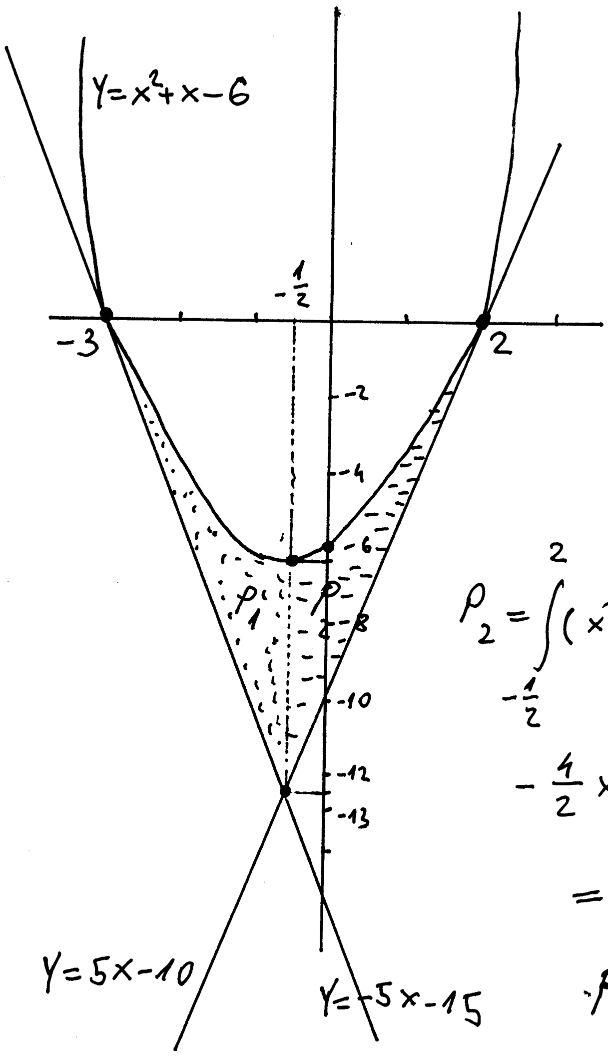
$y=-\frac{25}{2}=-12\frac{1}{2}$

$(1)-(2): \quad -10x-5=0$

$-10x=5$

$x=-\frac{1}{2}$

$\left(-\frac{1}{2}, -12\frac{1}{2}\right)$ je tačka presjeka pravih



$P=P_1+P_2$

$$P_1=\int_{-3}^{-\frac{1}{2}}\left(x^2+x-6-(-5x-15)\right)dx=\int_{-3}^{-\frac{1}{2}}(x^2+6x+9)dx=$$

$$=\frac{1}{3}x^3\Big|_{-3}^{-\frac{1}{2}}+\frac{6}{2}x^2\Big|_{-3}^{-\frac{1}{2}}+9x\Big|_{-3}^{-\frac{1}{2}}=\frac{1}{3}\left(-\frac{1}{8}+27\right)$$

$$+3\left(\frac{1}{4}-9\right)+9\left(-\frac{1}{2}+3\right)=\frac{1}{3}\cdot\frac{215}{8}+3\cdot\frac{-35}{4}$$

$$+9\cdot\frac{5}{2}=\frac{215}{24}-\frac{630}{24}+\frac{540}{24}=\frac{125}{24}$$

$$P_2=\int_{-\frac{1}{2}}^{2}\left(x^2+x-6-(5x-10)\right)dx=\int_{-\frac{1}{2}}^{2}(x^2-4x+4)\,dx=\frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{2}-$$

$$-\frac{4}{2}x^2\Big|_{-\frac{1}{2}}^{2}+4x\Big|_{-\frac{1}{2}}^{2}=\frac{1}{3}\left(8+\frac{1}{8}\right)-2\left(4-\frac{1}{4}\right)+4\left(2+\frac{1}{2}\right)=$$

$$=\frac{1}{3}\cdot\frac{65}{8}-2\cdot\frac{15}{4}+4\cdot\frac{5}{2}=\frac{65}{24}-\frac{180}{24}+\frac{240}{24}=\frac{125}{24}$$

$P=P_1+P_2=\frac{125}{24}+\frac{125}{24}=\frac{125}{12}$ tražena površina

(#) Izračunati površinu figure koju ograničavaju linije
$x = y^2 - 2y - 3$ i $y = 3 - 3x$

$y = 3 - 3x$
$3x = 3 - y$
$x = 1 - \frac{1}{3}y$

Rj. Nađimo presječnu tačku ovih linija

$$x = y^2 - 2y - 3$$
$$y = 3 - 3x$$

$$x = (3-3x)^2 - 2(3-3x) - 3$$
$$x = 9 - 18x + 9x^2 - 6 + 6x - 3$$
$$9x^2 - 13x = 0$$
$$x(9x - 13) = 0$$
$$x = 0 \quad \text{ili} \quad 9x = 13$$
$$x = \frac{13}{9}$$

$x = 0 \Rightarrow y = 3$

$x = \frac{13}{9} \Rightarrow y = 3 - \cancel{3} \cdot \frac{13}{\cancel{9}_3} = \frac{9}{3} - \frac{13}{3} = -\frac{4}{3}$

$A(0,3)$ i $B(\frac{13}{9}, -\frac{4}{3})$ su presječne tačke linija

$x = y^2 - 2y - 3$ je kriva oblika parabole C čije je tjeme $T(-\frac{D}{4a}, -\frac{b}{2a})$

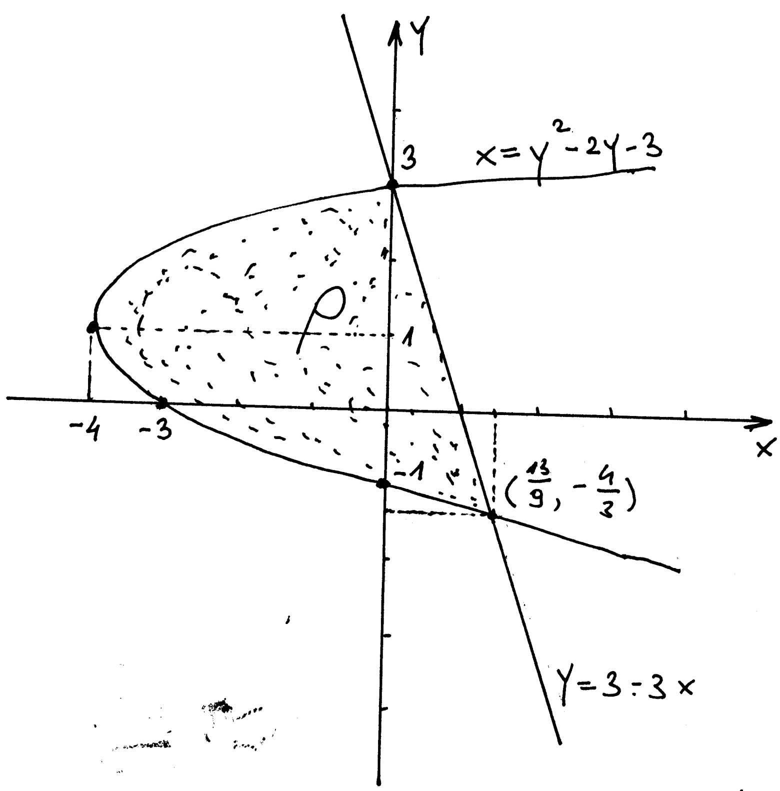
$-\frac{b}{2a} = -\frac{-2}{2} = 1$, $\quad D = 4 + 12 = 16$, $\quad -\frac{D}{4a} = -\frac{16}{4} = -4$

$T(1, -4)$

$$y_{1,2} = \frac{2 \pm 4}{2}$$
$$y_1 = \frac{-2}{2} = -1 \quad y_2 = \frac{6}{2} = 3$$

$M_1(0,-1)$ i $M_2(0,3)$ su presjek parabole sa y-osom

$x = y^2 - 2y - 3$
$y = 0 \Rightarrow x = -3$
$(-3,0)$ je presjek krive sa x-osom



$$P = \int_{-\frac{4}{3}}^{3} \left[\left(1 - \frac{1}{3}y\right) - (y^2 - 2y - 3)\right] dy = \int_{-\frac{4}{3}}^{3} \left(-y^2 + \frac{5}{3}y + 4\right) dy =$$

$$= -\frac{1}{3}y^3\Big|_{-\frac{4}{3}}^{3} + \frac{5}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{4}{3}}^{3} + 4y\Big|_{-\frac{4}{3}}^{3} =$$

$$= -\frac{1}{3}\left(27 + \frac{64}{27}\right) + \frac{5}{6}\left(9 - \frac{16}{9}\right) + 4\left(3 + \frac{4}{3}\right)$$

$$= -\frac{1}{3}\cdot\frac{793}{27} + \frac{5}{6}\cdot\frac{65}{9} + 4\cdot\frac{13}{3} =$$

$$= -\frac{793}{81} + \frac{325}{54} + \frac{52}{3} = \frac{-793\cdot 2 + 325\cdot 3 + 52\cdot 54}{162} = \frac{-1586 + 975 + 2808}{162}$$

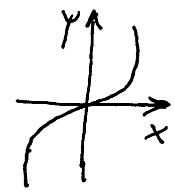
$$P = \frac{2197}{162} = 13\frac{91}{162}$$ tražena površina

7) Izračunati površinu figure koja je određena linijama $y=-x$, $y=\sqrt[3]{x}$, $y=3x-2$.

Rj.

Grafički nije teško predstaviti prave $y=-x$ i $y=3x-2$. Problem predstavlja kriva $y=\sqrt[3]{x}$.

Ako znamo da kriva $y=x^3$ izgleda ovako



Onda nije teško nacrtati krivu $x=y^3$ što je ekvivalentno sa $y=\sqrt[3]{x}$.

Pronađimo tačke presjeka datih krivih.

$$\begin{aligned} y&=-x\\ y&=3x-2\\ \hline -x&=3x-2\\ -4x&=-2\\ x&=\frac{1}{2} \Rightarrow y=-\frac{1}{2}\end{aligned}$$

$$\begin{aligned} y&=-x\\ y&=\sqrt[3]{x}\\ \hline y&=-x\\ y^3&=x\\ \hline -x^3&=x\\ x^3+x&=0\\ x(x^2+1)&=0\\ x=0 &\Rightarrow y=0\end{aligned}$$

$$\begin{aligned} y&=3x-2\\ y&=\sqrt[3]{x}\\ \hline \sqrt[3]{x}&=3x-2\\ (3x-2)^3&=x\\ 27x^3-3\cdot(3x)^2\cdot 2 &+ 3\cdot 3x\cdot(-2)^2+(-2)^3=x\\ 27x^3-54x^2+36x-8&=x\\ 27x^3-54x^2+35x-8&=0\end{aligned}$$

pokušajmo riješiti sistem na drugi način

$\sqrt[3]{x}=3x-2$

$\boxed{3x=y+2}$

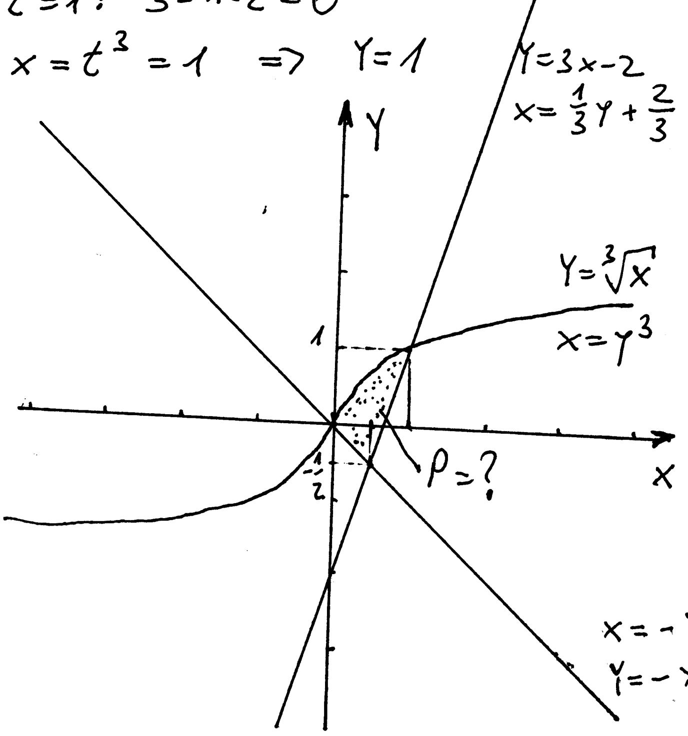
$x=t^3$

$3t^3-2=t$

$3t^3-t-2=0$

$t=1:\ 3-1-2=0$

$x=t^3=1 \Rightarrow y=1$



$$P=\int_{-\frac{1}{2}}^{0}\left[\left(\frac{1}{3}y+\frac{2}{3}\right)-(-y)\right]dy+\int_{0}^{1}\left[\left(\frac{1}{3}y+\frac{2}{3}\right)-y^3\right]dy=$$

$$=\int_{-\frac{1}{2}}^{0}\left(\frac{4}{3}y+\frac{2}{3}\right)dy+\int_{0}^{1}\left(-y^3+\frac{1}{3}y+\frac{2}{3}\right)dy=$$

$$=\frac{4}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{1}{2}}^{0}+\frac{2}{3}y\Big|_{-\frac{1}{2}}^{0}-\frac{1}{4}y^4\Big|_{0}^{1}+\frac{1}{3}\cdot\frac{1}{2}y^2\Big|_{0}^{1}$$

$$+\frac{2}{3}y\Big|_{0}^{1}=\frac{\cancel{2}}{3}\cdot\left(-\frac{1}{\cancel{4}_2}\right)+\frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}-\frac{1}{4}+\frac{1}{6}+$$

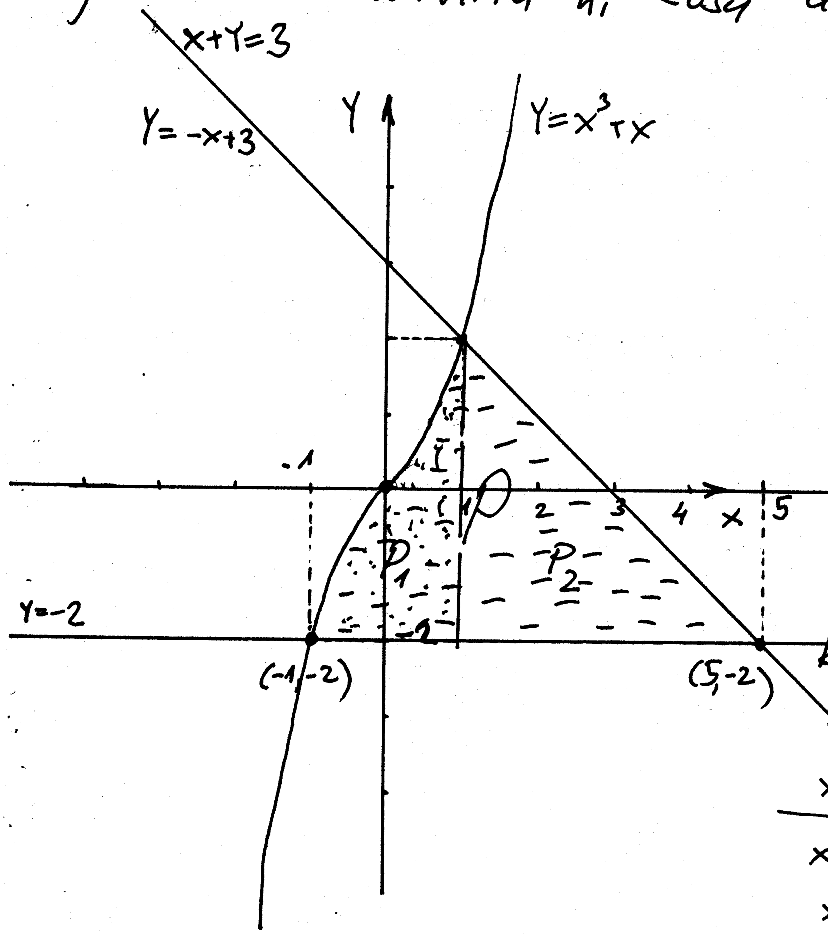
$$+\frac{2}{3}=-\cancel{\frac{1}{6}}+\frac{1}{3}-\frac{1}{4}+\cancel{\frac{1}{6}}+\frac{2}{3}=\frac{3}{4}$$

Izračunati površinu figure koja je određena linijama $y=-2$, $y=x^3+x$, $x+y=3$.

Rj. $y=-2$, $x+y=3$ su prave linije i njih nije teško nacrtati.
Problem za crtanje predstavlja kriva $y=x^3+x$.
Ispitajmo f-ju $y=x^3+x$. $\mathcal{D}: x\in\mathbb{R}$

$f(-x)=-x^3-x=-(x^3+x)$ f-ja je neparna

$A(0,0)$ je nula f-je i presjek sa y-osom
f-ja nema prekida $\Rightarrow$ f-ja nema vertikalnu asimptotu
f-ja nema horizontalnu ni kosu asimptotu



$y'=3x^2+1$ f-ja je uvijek pozitivna (raste za svako x)

f-ja nema ekstrem

$y''=6x$

x	$(0,+\infty)$
y''	$+$
y	$\cup$

$(0,0)$ je prevojna tačka

f-ja je ovog oblika

Nađimo tačke presjeka datih krivih.

$$\begin{array}{l} y=-2 \\ x+y=3 \\ \hline x-2=3 \\ x=5 \end{array}$$

$(5,-2)$ je tačka presjeka

$$\begin{array}{l} y=-2 \\ y=x^3+x \\ \hline -2=x^3+x \\ x^3+x+2=0 \end{array}$$

$x=-1$: $-1-1+2=0$

$$(x^3+x+2):(x+1)=x^2-x+2$$
$$\begin{array}{l} -\ \ x^3+x^2 \\ \hline -x^2+x+2 \\ -\ \ -x^2-x \\ \hline 2x+2 \\ 2x+2 \\ \hline // \ // \end{array}$$

$x^3+x+2=(x+1)\underbrace{(x^2-x+2)}_{>0\ \forall x}$

Rješenje jednačine $x^3+x+2=0$ je $x=-1$.
$(-1,-2)$ je tačka presjeka datih krivih

$$y = x^3 + x$$
$$x + y = 3$$

$$y = x^3 + x$$
$$y = -x + 3$$

$$-x + 3 = x^3 + x$$

$$x^3 + 2x - 3 = 0$$

$$x = 1: \quad 1^3 + 2\cdot 1 - 3 = 3 - 3 = 0$$

$$(x^3 + 2x - 3) : (x - 1) = x^2 + x + 3$$

$$\begin{array}{l} x^3 - x^2 \\ \hline x^2 + 2x - 3 \\ -\ x^2 - x \\ \hline 3x - 3 \\ -\ 3x - 3 \\ \hline = \quad = \end{array}$$

$$x^3 + 2x - 3 = \underbrace{(x^2 + x + 3)}_{>0\ \forall x}(x - 1)$$

$(1, 2)$ je presečna tačka krivih

$$P_1 = \int_{-1}^{1} \left[(x^3 + x) - (-2)\right] dx = \int_{-1}^{1} (x^3 + x + 2)\, dx = \frac{1}{4} x^4 \Big|_{-1}^{1} + \frac{1}{2} x^2 \Big|_{-1}^{1} + 2x \Big|_{-1}^{1} = 4$$

$$P_2 = \int_{1}^{5} \left[(-x + 3) - (-2)\right] dx = \int_{1}^{5} (-x + 5)\, dx = -\frac{x^2}{2} \Big|_{1}^{5} + 5x \Big|_{1}^{5} =$$

$$= -\frac{1}{2}(25 - 1) + 5 \cdot 4 = -\frac{1}{2} \cdot 24 + 20 = 20 - 12 = 8$$

$P = P_1 + P_2 = 8 + 4 = 12$ površina figure

(#) Izračunati površinu figure koju čine linije

$$y=(x-1)^2, \quad \frac{x^2}{1}-\frac{y^2}{2}=1.$$

Rj. Da bi odredili granice za računanje površine potrebno je grafički predstaviti ove dvije linije.

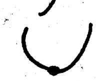
ispitajmo f-ju $y=(x-1)^2$

$D: x \in \mathbb{R}$

f-ja nije ni parna ni neparna

$f(0)=1$, $(0,1)$ je presjek sa y-osom

$(x-1)^2=0 \Rightarrow x=1$ $(1,0)$ je nula f-je

$y=(x-1)^2=x^2-2x+1 \Rightarrow$ f-ja je  oblika

Nađimo još tjeme f-je

$y' = 2x-2$

$y'=0 \Rightarrow 2x-2=0 \Rightarrow x=1$

$T(1,0)$

Prije nego grafički predstavimo liniju $y=(x-1)^2$ pronađimo u kojim tačkama siječe liniju $\frac{x^2}{1}-\frac{y^2}{2}=1$.

$$y=x^2-2x+1=(x-1)^2$$

$$2x^2-y^2=2$$

$$y^2=(x^2-2x+1)^2=$$
$$=(x^2-2x+1)(x^2-2x+1)=$$
$$=x^4-2x^3+x^2-2x^3+4x^2-2x+x^2-2x+1$$
$$=x^4-4x^3+6x^2-4x+1.$$

$$2x^2-y^2=2$$

$$2x^2-x^4+4x^3-6x^2+4x-1-2=0 \quad /\cdot(-1)$$

$$x^4-4x^3+4x^2-4x+3=0$$

Označimo ovo sa $g(x)=x^4-4x^3+4x^2-4x+3$

Kako je $g(1)=0 \Rightarrow g(x)$ je djeljivo sa $(x-1)$

$$(x^4-4x^3+4x^2-4x+3):(x-1)=x^3-3x^2+x-3$$

$$\begin{array}{l} \underline{-\;x^4-x^3} \\ \quad -3x^3+4x^2-4x+3 \\ \quad \underline{-\;-3x^3+3x^2} \\ \qquad x^2-4x+3 \\ \qquad \underline{-\;x^2-x} \\ \qquad\quad -3x+3 \\ \qquad\quad \underline{-3x+3} \\ \qquad\qquad = = \end{array}$$

$$g(x)=\underbrace{(x^3-3x^2+x-3)}_{g_1(x)}(x-1)$$

$g_1(0)=-3$

$g_1(1)=1-3+1-3=-4$

$g_1(2)=8-12+2-3=-5$

$g_1(3)=27-27+3-3 \Rightarrow g_1(x)$ je djeljivo sa $x-3$

$g_1(-2)=-8-12-2-3=-25$

$g_1(-1)=-1-3-1-3=-8$

$g_1(-3)=-27-27-3-3<0$

$$(x^3-3x^2+x-3):(x-3)=x^2+1$$

$$\begin{array}{l} \underline{-\;x^3-3x^2} \\ \qquad x-3 \\ \qquad \underline{-\;x-3} \\ \qquad\quad = = \end{array}$$

Prema tome $g(x)=(x^2+1)(x-3)(x-1)$

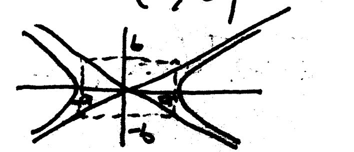
Za $x=3 \Rightarrow y=4$

Za $x=1 \Rightarrow y=0$

Presječne tačke krivih su $(3,4)$ i $(1,0)$

Krive oblika

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo HIPERBOLE i one su oblika

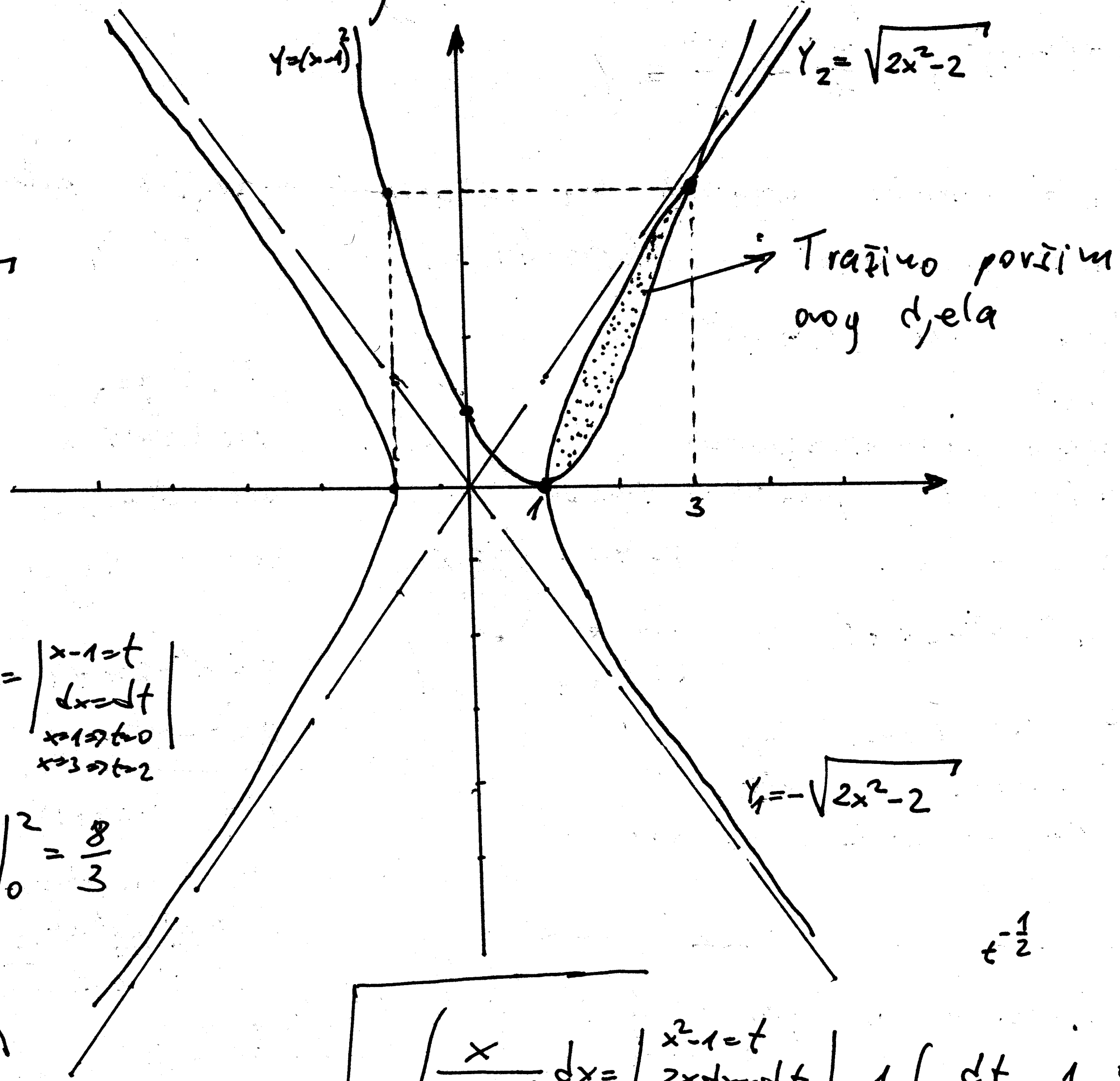


Nacrtajmo naše krive linije

$\sqrt{2} \approx 1{,}41$

$$y^2 = 2x^2 - 2$$

$$Y_{1,2} = \pm\sqrt{2x^2-2}$$



$$P_2 = \int_1^3 (x-1)^2\,dx = \left| \begin{matrix} x-1=t \\ dx=dt \\ x=1 \Rightarrow t=0 \\ x=3 \Rightarrow t=2 \end{matrix} \right|$$

$$= \int_0^2 t^2\,dt = \frac{1}{3}t^3\Big|_0^2 = \frac{8}{3}$$

$$t^{-\frac{1}{2}}$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} x^2-1=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \sqrt{t} + C = \sqrt{x^2-1} + C$$

$$P = \int_1^3 \Big(\underbrace{\sqrt{2x^2-2}}_{P_1} - \underbrace{(x-1)^2}_{P_2}\Big)\,dx$$

$$P_1 = \int_1^3 \sqrt{2}\cdot\sqrt{x^2-1}\,dx = \sqrt{2}\int_1^3 \frac{x^2-1}{\sqrt{x^2-1}}\,dx = \sqrt{2}\left(\int_1^3 \frac{x^2}{\sqrt{x^2-1}}\,dx - \int_1^3 \frac{dx}{\sqrt{x^2-1}}\right)$$

$$\int_1^3 x\cdot\frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} u=x & dv=\frac{x}{\sqrt{x^2-1}}\,dx \\ du=dx & v=\sqrt{x^2-1} \end{matrix} \right| = \underbrace{x\sqrt{x^2-1}\Big|_1^3}_{3\sqrt{8}-0} - \int_1^3 \sqrt{x^2-1}\,dx$$

$$\int_1^3 \frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}|\Big|_1^3 = \ln|3+\sqrt{8}|$$

$$\sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx = \sqrt{2}\cdot\underbrace{3\sqrt{8}}_{6\sqrt{2}} - \sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx - \sqrt{2}\ln(3+2\sqrt{2})$$

$$\int_1^3 \sqrt{2x^2-2}\,dx = 6 - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

$$P = P_1 - P_2 = \frac{10}{3} - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

tražena površina

Dio tablice integrala

1. $\int u^{a}\,du=\frac{u^{a+1}}{a+1}+C,\ \ a\neq-1.$

2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$
$=\ln|u|+C.$

3. $\int a^{u}\,du=\frac{a^{u}}{\ln a}+C;\ \int e^{u}\,du=$
$=e^{u}+C.$

4. $\int \sin u\,du=-\cos u+C.$

5. $\int \cos u\,du=\sin u+C.$

6. $\int \sec^{2}u\,du=\operatorname{tg}u+C.$

7. $\int \operatorname{cosec}^{2}u\,du=-\operatorname{ctg}u+C.$

8. $\int\frac{du}{u^{2}+a^{2}}=\frac{1}{a}\operatorname{arc\,tg}\frac{u}{a}+C.$

9. $\int\frac{du}{u^{2}-a^{2}}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int\frac{du}{\sqrt{a^{2}-u^{2}}}=\arcsin\frac{u}{a}+C.$

11. $\int\frac{du}{\sqrt{u^{2}+a}}=$
$=\ln\left|u+\sqrt{u^{2}+a}\right|+C.$